A Characterization of Weighted Fock Space Operators (Trends in Infinite Dimensional Analysis and Quantum Probability)

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A Characterization of Weighted Fock Space Operators

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Abstract

Let $\Gamma_\alpha(H_{\mathbf{C}})$ be a weighted Fock space over a complex Hilbert space $H_{\mathbf{C}}$ with weighted sequence α. In this paper we define S_α-transform of vectors in the weighted Fock space and then the vectors in $\Gamma_\alpha(H_{\mathbf{C}})$ and operators on the weighted Fock space are characterized on the basis of Bargmann–Segal space. As an application we discuss a regular property of solutions of normal-ordered differential equations.

1 Introduction

The white noise calculus initiated by Hida [12] has developed into an infinite dimensional analogue of Schwartz type distribution theory with wide applications ([13], [14], [23], [27], etc). The S-transforms ([1], [8], [9], [21] [33]) and the operator symbols ([3], [4], [19], [20], [26]) in white noise calculus are characterized as entire functions on an infinite dimensional vector space having particular growth rates. Since those characterizations depend heavily on the nuclearity of the space of test white noise functionals, elements in the (Boson) Fock space or bounded operators on the Fock space have not been characterized in a similar manner. Some partial results are found in [7].

Recently, in [10], the S-transforms of vectors in different Fock spaces are characterized by means of the Bargmann–Segal space ([24], [34], see also [2], [11]). The idea used in [10] was naturally extended to characterize the symbols of operators in several classes of operators on Fock space in [18], and the characterizations have been widely applied to study expansion theorems ([4], [27]) and (nonlinear white noise) differential equation which is a generalization of normal-ordered differential equations ([5], [6], [7], [16], [30], [31]) involving the quantum stochastic differential equation of Itô type formulated in [17] (see also [25], [32]). For white noise approach to quantum stochastic calculus we refer to [15], [28], [29].

Main purpose of this paper is to characterize vectors in weighted Fock spaces and the operators on the weighted Fock spaces on the basis of Bargmann–Segal space. This paper is organized as follows: In Section 2 we introduce the Bargmann–Segal space after [10]. In Section 3 we review the basic construction of riggings of Fock space (see [8], [21], [22]).

In Section 4 we define S_α-transform as a unitary isomorphism between the weighted Fock space and the Fock space, and characterize vectors in the weighted Fock space by means of S_α-transform. In Section 5 we define α-symbol of operators on weighted Fock space and its characterizations are investigated. In Section 6 we study Wick exponentials of operators on weighted Fock space. In Section 7 as an application we discuss a regular property of solutions of normal-ordered differential equations.

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2 Bargmann–Segal Space

Let K be a selfadjoint operator on $H = L^2(\mathbf{R}, dt)$ such that the Schwartz space $\mathcal{S}(\mathbf{R})$ is densely and continuously imbedded in $\mathrm{Dom}(K^p)$ for any $p \geq 0$ and is kept invariant under K. We assume that $K \geq 1$.

For $p \in \mathbf{R}$ we put

$$|\xi|_{K,p} = |K^p \xi|_0, \qquad \xi \in H,$$

where $|\cdot|_0$ is the norm on H generated by the usual inner product $\langle \cdot, \cdot \rangle$. Then, for $p \geq 0$, the set $D_p = \{\xi \in H; |\xi|_{K,p} < \infty\}$ becomes a Hilbert space with norm $|\cdot|_{K,p}$. While, for $p \leq 0$, D_{-p} denotes the completion of H with respect to the norm $|\cdot|_{K,-p}$. Note that D_p and D_{-p} are dual each other. Then we have

$$D \equiv \operatorname*{proj\,lim}_{p \to \infty} D_p \subset H \subset D^* \cong \operatorname*{ind\,lim}_{p \to \infty} D_{-p},$$

where $\cong$ stands a topological isomorphism. In particular, by using the harmonic oscillator $A = -d^2/dt^2 + t^2 + 1$, we construct the Gelfand triple:

$$\mathcal{S}(\mathbf{R}) \subset H \subset \mathcal{S}'(\mathbf{R}), \tag{2.1}$$

where $\mathcal{S}'(\mathbf{R})$ the space of tempered distributions. From now on, for simple notation, we use $E \equiv \mathcal{S}(\mathbf{R})$ and $E^* \equiv \mathcal{S}'(\mathbf{R})$. The canonical bilinear form on $E^* \times E$ is denoted by the symbol $\langle \cdot, \cdot \rangle$ again.

By the Bochner–Minlos theorem, there exists a probability measure $\mu_{1/2}$ on E^* such that whose characteristic function is given by

$$\exp\left\{-\frac{1}{4}\langle \xi, \xi \rangle\right\} = \int_{E^*} e^{i\langle x, \xi \rangle} \mu_{1/2}(dx), \qquad \xi \in E.$$

For a topological space $\mathfrak{X}$, $\mathfrak{X}_{\mathbf{C}}$ denotes the complexification of $\mathfrak{X}$. Define a probability measure ν on $E^*_{\mathbf{C}} = E^* + iE^*$ in such a way that

$$\nu(dz) = \mu_{1/2}(dx) \times \mu_{1/2}(dy), \qquad z = x + iy, \quad x, y \in E^*.$$

Following Hida [13] the probability space $(E^*_{\mathbf{C}}, \nu)$ is called the *complex Gaussian space* associated with (2.1).

The *Bargmann–Segal space* [10], denoted by $\mathcal{E}^2(\nu)$, is by definition the space of entire functions $g : H_{\mathbf{C}} \to \mathbf{C}$ such that

$$\| g \|_{\mathcal{E}^2(\nu)}^2 \equiv \sup_{P\in\mathcal{P}} \int_{E_{\mathbf{C}}^*} |g(Pz)|^2 \nu(dz) < \infty,$$

where $\mathcal{P}$ is the set of all finite rank projections on H with range contained in E. Note that $P \in \mathcal{P}$ is naturally extended to a continuous operator from $E_{\mathbf{C}}^*$ into $H_{\mathbf{C}}$ (in fact into $E_{\mathbf{C}}$), which is denoted by the same symbol. The Bargmann–Segal space $\mathcal{E}^2(\nu)$ is a Hilbert space with norm $\|\cdot\|_{\mathcal{E}^2(\nu)}$. Let $\Gamma(H_{\mathbf{C}})$ be the (Boson) Fock space over the complex Hilbert space $H_{\mathbf{C}}$ (see §3). For $\phi = (f_n)_{n=0}^{\infty} \in \Gamma(H_{\mathbf{C}})$ define

$$J\phi(\xi) = \sum_{n=0}^{\infty} \langle \xi^{\otimes n}, f_n \rangle, \qquad \xi \in H_{\mathbf{C}},$$

where the right hand side converges uniformly on each bounded subset of $H_{\mathbf{C}}$. Hence $J\phi$ becomes an entire function on $H_{\mathbf{C}}$. Moreover, it is known (e.g., [10], [11], [18]) that J becomes a unitary isomorphism from $\Gamma(H_{\mathbf{C}})$ onto $\mathcal{E}^2(\nu)$ and is called the *duality transform*.

3 Riggings of Fock Space

Let H be a Hilbert space with norm $|\cdot|$. For $n \geq 0$ let $H^{\widehat{\otimes} n}$ be the n-fold symmetric tensor power of H and their norms are denoted by the common symbol $|\cdot|$. Given a positive sequence $\alpha = \{\alpha(n)\}_{n=0}^{\infty}$ we put

$$\Gamma_\alpha(H) = \left\{ \phi = (f_n)_{n=0}^{\infty};\, f_n \in H^{\widehat{\otimes} n},\, \| \phi \|_+^2 \equiv \sum_{n=0}^{\infty} n!\, \alpha(n) \,| f_n |^2 < \infty \right\}.$$

Then $\Gamma_\alpha(H)$ becomes a Hilbert space and is called a *weighted Fock space* with weighted sequence α. The Boson Fock space $\Gamma(H)$ is the special case of $\alpha(n) \equiv 1$.

For a weight sequence $\alpha = \{\alpha(n)\}$ we consider the following four conditions:

(A1) $\alpha(0) = 1$ and $\inf_{n\geq 0} \alpha(n)\sigma^n > 0$ for some $\sigma \geq 1$;

(A2) $\lim_{n\to\infty} \left(\dfrac{\alpha(n)}{n!} \right)^{1/n} = 0$;

(A3) α is equivalent to a positive sequence γ such that $\{\gamma(n)/n!\}$ is log-concave;

(A4) α is equivalent to another positive sequence γ such that $\{(n!\gamma(n))^{-1}\}$ is log-concave.

The generating function of $\{\alpha(n)\}$ is defined by

$$G_\alpha(t) = \sum_{n=0}^{\infty} \frac{\alpha(n)}{n!} t^n.$$

By conditions (A1) and (A2), $G_\alpha(t)$ is entire. Put

$$\widetilde{G}_\alpha(t) = \sum_{n=0}^{\infty} \frac{n^{2n}}{n!\alpha(n)} \left\{ \inf_{s>0} \frac{G_\alpha(s)}{s^n} \right\} t^n.$$

Then it is known [1] that (A3) is necessary and sufficient condition for $G_\alpha(t)$ to have positive radius of convergence $R_\alpha > 0$.

From now on we always assume that a weight sequence $\alpha = \{\alpha(n)\}$ satisfies conditions (A1)–(A4).

Lemma 1 [1] *For a weight sequence $\alpha = \{\alpha(n)\}$, we have*

(1) *There exists a constant $C_{1\alpha} > 0$ such that*

$$\alpha(n)\alpha(m) \le C_{1\alpha}^{n+m}\alpha(n+m), \qquad n, m = 0, 1, 2, \cdots.$$

(2) *There exists a constant $C_{2\alpha} > 0$ such that*

$$\alpha(n+m) \le C_{2\alpha}^{n+m}\alpha(n)\alpha(m), \qquad n, m = 0, 1, 2, \cdots.$$

(3) *There exists a constant $C_{3\alpha} > 0$ such that*

$$\alpha(m) \le C_{3\alpha}^{n}\alpha(n), \qquad m \le n.$$

Now, we construct a chain of weighted Fock spaces over the rigged Hilbert spaces. For simplicity we set

$$\mathfrak{D}_{\alpha,p} = \Gamma_\alpha(D_{p,\mathbf{C}}), \qquad p \ge 0.$$

For $p \ge 0$, by definition, the norm of $\mathfrak{D}_{\alpha,p}$ is given by

$$\| \phi \|_{K,p,+}^2 = \sum_{n=0}^{\infty} n!\alpha(n) \, | f_n |_{K,p}^2, \quad \phi = (f_n), \quad f_n \in D_{p,\mathbf{C}}^{\widehat{\otimes} n}.$$

Then for any $0 \le p \le q$ we naturally come to

$$\begin{aligned} \mathfrak{D}_\alpha \equiv \operatorname*{proj\,lim}_{p\to\infty} \mathfrak{D}_{\alpha,p} \subset \cdots \subset \mathfrak{D}_{\alpha,q} \subset \cdots \subset \mathfrak{D}_{\alpha,p} \subset \cdots \\ \cdots \subset \Gamma(H_\mathbf{C}) \subset \cdots \subset \mathfrak{D}_{1/\alpha,-p} \subset \cdots \subset \mathfrak{D}_{1/\alpha,-q} \subset \cdots \subset \mathfrak{D}_\alpha^*, \end{aligned}$$

where for $p \ge 0$, $\mathfrak{D}_{1/\alpha,-p} = \Gamma_{1/\alpha}(D_{-p,\mathbf{C}})$. In particular, by using the harmonic oscillator A, we construct the following:

$$\mathfrak{W}_\alpha \subset \mathfrak{W}_{\alpha,p} \subset \Gamma(H_\mathbf{C}) \subset \mathfrak{W}_{1/\alpha,-p} \subset \mathfrak{W}_\alpha^*, \qquad p \ge 0$$

which is referred to as the *Cochran–Kuo–Sengupta space* with weight sequence $\alpha = \{\alpha(n)\}$. The one corresponding to $\alpha(n) = \widetilde{\beta}(n) = (n!)^\beta$, $0 \le \beta < 1$, is called the *Kondratiev–Streit space* [21] and is denoted by

$$\mathfrak{W}_{\widetilde{\beta}} = (E)_\beta, \qquad \widetilde{\beta}(n) = (n!)^\beta, \quad 0 \le \beta < 1.$$

The canonical complex bilinear form on $\mathfrak{W}_\alpha^* \times \mathfrak{W}_\alpha$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. Then

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \qquad \Phi = (F_n) \in \mathfrak{W}_\alpha^*, \quad \phi = (f_n) \in \mathfrak{W}_\alpha,$$

and it holds that

$$|\langle\langle \Phi, \phi \rangle\rangle| \leq \| \Phi \|_{A,-p,-} \| \phi \|_{A,p,+},$$

where

$$\| \Phi \|_{A,-p,-}^2 = \sum_{n=0}^{\infty} \frac{n!}{\alpha(n)} | F_n |_{A,-p}^2, \qquad \Phi = (F_n).$$

Now, we define a linear operator Γ_α from the weighted Fock space $\Gamma_\alpha(H_{\mathbf{C}})$ into the Fock space $\Gamma(H_{\mathbf{C}})$ by

$$\Gamma_\alpha(\phi) = (\sqrt{\alpha_n} f_n), \qquad \phi = (f_n) \in \Gamma_\alpha(H_{\mathbf{C}}).$$

Then it is obvious that Γ_α is a unitary isomorphism between $\Gamma_\alpha(H_{\mathbf{C}})$ and $\Gamma(H_{\mathbf{C}})$. In fact, for any $\phi = (f_n), \psi = (g_n) \in \Gamma_\alpha(H_{\mathbf{C}})$ we have

$$\left\langle\left\langle \Gamma_\alpha(\phi), \overline{\Gamma_\alpha(\psi)} \right\rangle\right\rangle_{\Gamma(H_{\mathbf{C}})} = \sum_{n=0}^{\infty} n! \alpha_n \langle f_n, \overline{g_n} \rangle = \langle\langle \phi, \overline{\psi} \rangle\rangle_{\Gamma_\alpha(H_{\mathbf{C}})}.$$

4 S_α-transform

For any positive sequence $\alpha = \{\alpha(n)\}$ and for each $\xi \in E_{\mathbf{C}}$, we put

$$\phi_{\alpha,\xi} = \left(\sqrt{\alpha(0)}, \sqrt{\alpha(1)}\xi, \frac{\sqrt{\alpha(2)}\xi^{\otimes 2}}{2!}, \cdots, \frac{\sqrt{\alpha(n)}\xi^{\otimes n}}{n!}, \cdots \right).$$

Then for any $\xi \in E_{\mathbf{C}}$ we have

$$\|\phi_{\alpha,\xi}\|_0^2 = \sum_{n=0}^{\infty} n! \frac{\alpha(n)}{n!^2} |\xi|_0^2 = G_\alpha(|\xi|_0^2),$$

where $\| \cdot \|_0$ is the norm on $\Gamma(H_{\mathbf{C}})$, and for any $p \geq 0$

$$\|\phi_{1/\alpha,\xi}\|_{K,p,+}^2 = \sum_{n=0}^{\infty} n! \alpha(n) \frac{1}{n!^2 \alpha(n)} |\xi|_{K,p}^2 = e^{|\xi|_{K,p}^2}.$$

Therefore, for any $\xi \in E_{\mathbf{C}}$, $\phi_{\alpha,\xi} \in \Gamma(H_{\mathbf{C}})$ and $\phi_{1/\alpha,\xi} \in \mathfrak{D}_\alpha$. Moreover, it can be shown that $\{\phi_{\alpha,\xi}\,;\, \xi \in E_{\mathbf{C}}\}$ and $\{\phi_{1/\alpha,\xi}\,;\, \xi \in E_{\mathbf{C}}\}$ span dense subspaces of $\Gamma(H_{\mathbf{C}})$ and $\mathfrak{D}_\alpha$, respectively.

For $\Phi \in \Gamma(H_{\mathbf{C}})$, the $\mathbf{C}$-valued function $S_\alpha\Phi$ defined by

$$S_\alpha\Phi(\xi) = \langle\langle \Phi, \phi_{\alpha,\xi} \rangle\rangle, \qquad \xi \in E_{\mathbf{C}}$$

is called the S_α*-transform* of Φ. Similarly, for $\Psi \in \mathfrak{D}_\alpha^*$, the $S_{1/\alpha}$*-transform* of Ψ is defined by

$$S_{1/\alpha}\Psi(\xi) = \langle\langle \Psi, \phi_{1/\alpha,\xi} \rangle\rangle, \qquad \xi \in E_{\mathbf{C}}.$$

Then $\Phi \in \Gamma(H_{\mathbf{C}})$ and $\Psi \in \mathfrak{D}_\alpha^*$ are uniquely specified by the S_α-transform and $S_{1/\alpha}$-transform, respectively. Let $p \geq 0$. Then for each $\Phi = (f_n)_{n=0}^\infty \in \mathfrak{D}_{\alpha,p}$ and $\Psi = (g_n)_{n=0}^\infty \in \mathfrak{D}_{1/\alpha,-p}$, $S_\alpha\Phi$ and $S_{1/\alpha}\Psi$ can be extended to $D_{-p,\mathbf{C}}$ and $D_{p,\mathbf{C}}$, respectively. Moreover, we have

$$S_\alpha\Phi(z) = \sum_{n=0}^\infty \sqrt{\alpha(n)} \left\langle z^{\otimes n}, f_n \right\rangle, \qquad z \in D_{-p,\mathbf{C}}$$

and

$$S_{1/\alpha}\Psi(z) = \sum_{n=0}^\infty \frac{1}{\sqrt{\alpha(n)}} \left\langle z^{\otimes n}, g_n \right\rangle, \qquad z \in D_{p,\mathbf{C}},$$

where the right hand sides converge uniformly on each bounded subset of $D_{-p,\mathbf{C}}$ and $D_{p,\mathbf{C}}$, respectively. Therefore, $S_\alpha\Phi$ and $S_{1/\alpha}\Psi$ become entire functions on $D_{-p,\mathbf{C}}$ and $D_{p,\mathbf{C}}$, respectively. Moreover, it is easily checked by definition that $S_\alpha\Phi(K^p\cdot) \in \mathcal{E}^2(\nu)$ and $S_{1/\alpha}\Psi(K^{-p}\cdot) \in \mathcal{E}^2(\nu)$.

Proposition 2 *The* S_α*-transform is a unitary isomorphism between* $\Gamma_\alpha(H_{\mathbf{C}})$ *and* $\mathcal{E}^2(\nu)$.

Proof. It is easily show that

$$S_\alpha = J \circ \Gamma_\alpha : \Gamma_\alpha(H_{\mathbf{C}}) \longrightarrow \Gamma(H_{\mathbf{C}}) \longrightarrow \mathcal{E}^2(\nu).$$

Since Γ_α and J are unitary isomorphisms, the proof follows. ■

Theorem 3 *Let* $p \geq 0$ *and* g *be a* **C***-valued function defined on* $E_{\mathbf{C}}$. *Then*

(1) g *is the* S_α*-transform of some* $\Phi \in \mathfrak{D}_{\alpha,p}$ *if and only if* g *can be extended to a continuous function on* $D_{-p,\mathbf{C}}$ *and* $g \circ K^p \in \mathcal{E}^2(\nu)$.

(2) g *is the* $S_{1/\alpha}$*-transform of some* $\Phi \in \mathfrak{D}_{1/\alpha,-p}$ *if and only if* g *can be extended to a continuous function on* $D_{p,\mathbf{C}}$ *and* $g \circ K^{-p} \in \mathcal{E}^2(\nu)$.

Proof. Since the proof of (2) is similar to the proof of (1), we only prove (1) by simply modified arguments used in [18]. Let g be a **C**-valued continuous function defined on $D_{-p,\mathbf{C}}$ such that $g \circ K^p \in \mathcal{E}^2(\nu)$. In fact, g is entire on $D_{-p,\mathbf{C}}$ since K^p is an isometry from $H_{\mathbf{C}}$ onto $D_{-p,\mathbf{C}}$. By the duality transform there exists $(f_n) \in \Gamma(H_{\mathbf{C}})$ such that

$$g \circ K^p(z) = \sum_{n=0}^\infty \left\langle z^{\otimes n}, f_n \right\rangle, \qquad z \in H_{\mathbf{C}}.$$

Then, changing variables, we have

$$g(\xi) = \sum_{n=0}^\infty \left\langle (K^{-p})^{\otimes n} f_n, \xi^{\otimes n} \right\rangle, \qquad \xi \in D_{-p,\mathbf{C}}.$$

Define $\Phi = (1/\sqrt{\alpha(n)}(K^{-p})^{\otimes n} f_n)$. Then by definition $\Phi \in \mathfrak{D}_{\alpha,p}$ and $S_\alpha\Phi(\xi) = g(\xi)$ for $\xi \in E_{\mathbf{C}}$, i.e., g is the S_α-transform of $\Phi \in \mathfrak{D}_{\alpha,p}$. The converse assertion is obvious. ■

During the above proof we have established the following

Proposition 4 *Let $p \geq 0$ and let $\Phi \in \mathfrak{D}_{\alpha,p}$, $\Psi \in \mathfrak{D}_{1/\alpha,-p}$. Then we have*

(1) *$S_\alpha \Phi$ admits a continuous extension to $D_{-p,\mathbf{C}}$ and $S_\alpha \Phi \circ K^p \in \mathcal{E}^2(\nu)$. Moreover,*

$$\| \Phi \|_{K,p,+} = \| S_\alpha \Phi \circ K^p \|_{\mathcal{E}^2(\nu)} .$$

(2) *$S_{1/\alpha} \Psi$ admits a continuous extension to $D_{p,\mathbf{C}}$ and $S_{1/\alpha} \Psi \circ K^p \in \mathcal{E}^2(\nu)$. Moreover,*

$$\| \Psi \|_{K,-p,-} = \left\| S_{1/\alpha} \Psi \circ K^{-p} \right\|_{\mathcal{E}^2(\nu)} .$$

By Theorem 3, the following corollary is obvious

Corollary 5 *Let g be a $\mathbf{C}$-valued function defined on $E_\mathbf{C}$. Then*

(1) *g is the S_α-transform of some $\Phi \in \mathfrak{D}_\alpha$ if and only if for any $p \geq 0$, g can be extended to a continuous function on $D_{-p,\mathbf{C}}$ and $g \circ K^p \in \mathcal{E}^2(\nu)$.*

(2) *g is the $S_{1/\alpha}$-transform of some $\Phi \in \mathfrak{D}_\alpha^*$ if and only if there exists $p \geq 0$ such that g can be extended to a continuous function on $D_{p,\mathbf{C}}$ and $g \circ K^{-p} \in \mathcal{E}^2(\nu)$.*

In the case of $\alpha \equiv 1$, the S_α-transform is called the *S-transform* (see [14], [23], [27]). For each $\Phi \in \mathfrak{W}_\alpha^*$, the S-transform $F = S\Phi$ possesses the following properties:

(F1) for each $\xi, \eta \in E_\mathbf{C}$, the function $z \mapsto F(z\xi + \eta)$ is entire holomorphic on $\mathbf{C}$;

(F2) there exist $C \geq 0$ and $p \geq 0$ such that

$$|F(\xi)|^2 \leq C G_\alpha(|\xi|_{A,p}^2), \qquad \xi \in E_\mathbf{C}.$$

The converse assertion is also true. This famous characterization theorem for S-transform was first proved for the Hida–Kubo–Takenaka space by Potthoff and Streit [33]. The following result is due to Cochran, Kuo and Sengupta [8].

Theorem 6 *Let F be a $\mathbf{C}$-valued function on $E_\mathbf{C}$. Then F is the S-transform of some $\Phi \in \mathfrak{W}_\alpha^*$ if and only if F satisfies conditions (F1) and (F2). In that case, for any $q > 1/2$ with $\| A^{-q} \|_{\mathrm{HS}}^2 < R_\alpha$ we have*

$$\| \Phi \|_{A,-(p+q),-}^2 \leq C \widetilde{G}_\alpha(\|A^{-q}\|_{\mathrm{HS}}^2).$$

5 Operators on Weighted Fock Space

Let $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ be the space of all continuous linear operators from a locally convex space $\mathfrak{X}$ into another locally convex space $\mathfrak{Y}$. Then a continuous linear operator $\Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*)$ is called a *generalized operator* (or *white noise operator*). Note that $\mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha)$ and $\mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{D}_{\alpha,p})$ are subspaces of $\mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*)$. Moreover, by duality, $\mathcal{L}(\mathfrak{W}_\alpha^*, \mathfrak{W}_\alpha^*)$ is isomorphic to $\mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha)$. A general theory for generalized operators has been extensively developed in [4], [27], [29].

The $1/\alpha$-*symbol*, which is an operator version of the $S_{1/\alpha}$-transform, of a generalized operator $\Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*)$ is defined as a complex valued function on $E_{\mathbf{C}} \times E_{\mathbf{C}}$ by

$$\widehat{\Xi}^{1/\alpha}(\xi, \eta) = \langle\langle \Xi\phi_\xi, \phi_{1/\alpha,\eta} \rangle\rangle, \qquad \xi, \eta \in E_{\mathbf{C}},$$

where $\phi_\xi = \phi_{1,\xi}$. In the case of $\alpha \equiv 1$, $\widehat{\Xi}^{1/\alpha}$ is denoted by $\widehat{\Xi}$ which is called the *symbol* of Ξ. Every generalized operator is uniquely determined by its symbol. By the definitions, we have the following relations:

$$\widehat{\Xi}^{1/\alpha}(\xi, \eta) = S_{1/\alpha}(\Xi\phi_\xi)(\eta) = S(\Xi^*\phi_{1/\alpha,\eta})(\xi), \qquad \xi, \eta \in E_{\mathbf{C}}.$$

As is easily verified, the symbol $\Theta = \widehat{\Xi}$ of a generalized operator $\Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*)$ possesses the following properties:

(O1) for any $\xi, \xi_1, \eta, \eta_1 \in E_{\mathbf{C}}$ the function $(z, w) \mapsto \Theta(z\xi + \xi_1, w\eta + \eta_1)$ is entire holomorphic on $\mathbf{C} \times \mathbf{C}$;

(O2) there exist constant numbers $C \geq 0$ and $p \geq 0$ such that

$$|\Theta(\xi, \eta)|^2 \leq C G_\alpha(|\,\xi\,|_{A,p}^2) G_\alpha(|\,\eta\,|_{A,p}^2), \qquad \xi, \eta \in E_{\mathbf{C}}.$$

As in the case of S-transform, the characterization theorem for symbols, which was first proved by Obata for the Hida–Kubo–Takenaka space, is a significant consequence of white noise theory. The characterization in the case of CKS–space was proved in [4].

Theorem 7 *A function* $\Theta : E_{\mathbf{C}} \times E_{\mathbf{C}} \to \mathbf{C}$ *is the symbol of a white noise operator* $\Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*)$ *if and only if* Θ *satisfies conditions (O1) and (O2). In that case*

$$\|\,\Xi\phi\,\|_{A,-(p+q),-}^2 \leq C\widetilde{G}_\alpha^2(\|\,A^{-q}\,\|_{\mathrm{HS}}^2)\,\|\,\phi\,\|_{A,p+q,+}^2, \qquad \phi \in \mathfrak{W}_\alpha,$$

where $q > 1/2$ *is taken as* $\|\,A^{-q}\,\|_{\mathrm{HS}}^2 < R_\alpha$.

We now study the characterization theorem for α-symbols of operators on weighted Fock spaces. For the characterization theorem for symbols of operators on Fock spaces we refer to [18]. Let $p \geq 0$. Then it is easily shown that for each $\Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{D}_{\alpha,p})$ the α-symbol $\widehat{\Xi}^\alpha$ of Ξ is well-defined and $\widehat{\Xi}^\alpha$ is extended to an entire function on $E_{\mathbf{C}} \times D_{-p,\mathbf{C}}$.

Theorem 8 *Let* $p \geq 0$ *and let* Θ *be a complex valued function defined on* $E_{\mathbf{C}} \times E_{\mathbf{C}}$. *Then there exists* $\Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{D}_{\alpha,p})$ *such that* $\Theta = \widehat{\Xi}^\alpha$ *if and only if*

(i) Θ *can be extended to an entire function on* $E_{\mathbf{C}} \times D_{-p,\mathbf{C}}$;

(ii) *there exist* $q \geq 0$ *and* $C \geq 0$ *such that*

$$\|\Theta(\xi, K^p\cdot)\|_{\mathcal{E}^2(\nu)}^2 \leq C G_\alpha(|\xi|_{A,q}^2), \qquad \xi \in E_{\mathbf{C}}.$$

Proof. For the proof we use similar arguments used in [18]. Suppose that there exists $\Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{D}_{\alpha,p})$ such that $\Theta = \widehat{\Xi}^\alpha$. Then condition (i) is obvious and there exists $q \geq 0$ such that $\Xi \in \mathcal{L}(\mathfrak{W}_{\alpha,q}, \mathfrak{D}_{\alpha,p})$. Hence there exists $C \geq 0$ such that

$$\|\Xi\phi\|_{K,p,+} \leq \sqrt{C}\|\phi\|_{A,q,+}, \qquad \phi \in \mathfrak{W}_{\alpha,q}.$$

Therefore, we have

$$\|\Theta(\xi, K^p\cdot)\|^2_{\mathcal{E}^2(\nu)} = \|\Xi\phi_\xi\|^2_{K,p,+} \leq C\|\phi_\xi\|^2_{A,q,+} = CG_\alpha(|\xi|^2_{A,q}).$$

Conversely, suppose that conditions (i) and (ii) are satisfied. Let $\xi \in E_{\mathbf{C}}$ be fixed and define a function $F_\xi : D_{-p,\mathbf{C}} \to \mathbf{C}$ by $F_\xi(\eta) = \Theta(\xi, \eta)$, $\eta \in D_{-p,\mathbf{C}}$. Then by (ii), $F_\xi(K^p\cdot) \in \mathcal{E}^2(\nu)$. Hence by Theorem 3, there exists $\Phi_\xi \in \mathfrak{D}_{\alpha,p}$ such that $S_\alpha(\Phi_\xi) = F_\xi$ and

$$\|\Phi_\xi\|^2_{K,p,+} = \|F_\xi \circ K^p\|^2_{\mathcal{E}^2(\nu)} = \|\Theta(\xi, K^p\cdot)\|^2_{\mathcal{E}^2(\nu)} \leq CG_\alpha(|\xi|^2_{A,q}).$$

Now, fix $\phi \in \mathfrak{D}_{1/\alpha,-p}$ and define a function $G_\phi : E_{\mathbf{C}} \to \mathbf{C}$ by

$$G_\phi(\xi) = \langle\langle \phi, \Phi_\xi \rangle\rangle, \qquad \xi \in E_{\mathbf{C}}.$$

Then we can easily show that G_ϕ satisfies conditions (F1) and (F2). In fact,

$$|G_\phi(\xi)|^2 \leq \|\phi\|^2_{K,-p,-}\|\Phi_\xi\|^2_{K,p,+} \leq C\|\phi\|^2_{K,-p,-}G_\alpha(|\xi|^2_{A,q}).$$

Therefore, by Theorem 6, there exists $\Psi_\phi \in \mathfrak{W}^*_\alpha$ such that

$$S(\Psi_\phi)(\xi) = G_\phi(\xi) = \langle\langle \phi, \Phi_\xi \rangle\rangle, \qquad \xi \in E_{\mathbf{C}}.$$

Moreover, we have

$$\| \Psi_\phi \|^2_{A,-(q+q'),-} \leq C\widetilde{G}_\alpha(\| A^{-q'} \|^2_{\mathrm{HS}})\| \phi \|^2_{K,-p,-} \tag{5.2}$$

for some $q' > 1/2$ with $\| A^{-q'} \|^2_{\mathrm{HS}} < R_\alpha$. Define a linear operator $\Xi^* : \mathfrak{D}_{1/\alpha,-p} \to \mathfrak{W}^*_\alpha$ by $\Xi^*\phi = \Psi_\phi$, $\phi \in \mathfrak{D}_{1/\alpha,-p}$. It then follows from (5.2) that $\Xi^* \in \mathcal{L}(\mathfrak{D}_{1/\alpha,-p}, \mathfrak{W}^*_\alpha)$. Then it is obvious that Θ is the α-symbol of $\Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{D}_{\alpha,p})$ (the adjoint of Ξ^*). ∎

By the similar arguments used in the proof of Theorem 8, we have

Theorem 9 *Let $p \geq 0$ and let Θ be a complex valued function defined on $E_{\mathbf{C}} \times E_{\mathbf{C}}$. Then there exists $\Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{D}_{1/\alpha,-p})$ such that $\Theta = \widehat{\Xi}^{1/\alpha}$ if and only if*

(i) *Θ can be extended to an entire function on $E_{\mathbf{C}} \times D_{p,\mathbf{C}}$;*

(ii) *there exist $q \geq 0$ and $C \geq 0$ such that*

$$\|\Theta(\xi, K^{-p}\cdot)\|^2_{\mathcal{E}^2(\nu)} \leq CG_\alpha(|\xi|^2_{A,q}), \qquad \xi \in E_{\mathbf{C}}.$$

For each $\kappa \in D_{p,\mathbf{C}}^{\otimes l} \otimes (E_{\mathbf{C}}^{\otimes m})^*$, let

$$\Theta(\xi,\eta) = \langle \kappa,\, \eta^{\otimes l} \otimes \xi^{\otimes m} \rangle \sum_{n=0}^{\infty} \frac{\sqrt{\alpha(n+l)}}{n!} \langle \xi,\, \eta \rangle^n, \qquad \xi,\eta \in E_{\mathbf{C}}. \tag{5.3}$$

Then it is obvious that Θ can be extended to an entire function on $E_{\mathbf{C}} \times D_{-p,\mathbf{C}}$.

For $\kappa \in ({E_{\mathbf{C}}}^{\otimes(l+m)})^*$ and $f \in {E_{\mathbf{C}}}^{\otimes(n+m)}$ the (right m-) contraction of a tensor product is defined by

$$\kappa \otimes_m f = \sum_{\mathbf{j},\mathbf{k}} \left(\sum_{\mathbf{i}} \langle \kappa,\, e(\mathbf{j}) \otimes e(\mathbf{i}) \rangle \langle f,\, e(\mathbf{k}) \otimes e(\mathbf{i}) \rangle \right) e(\mathbf{j}) \otimes e(\mathbf{k}),$$

where

$$e(\mathbf{i}) = e_{i_1} \otimes \cdots \otimes e_{i_m}, \quad e(\mathbf{j}) = e_{j_1} \otimes \cdots \otimes e_{j_l}, \quad e(\mathbf{k}) = e_{k_1} \otimes \cdots \otimes e_{k_n},$$

which form orthonormal bases of ${H_{\mathbf{C}}}^{\otimes m}$, ${H_{\mathbf{C}}}^{\otimes l}$, ${H_{\mathbf{C}}}^{\otimes n}$, respectively. We need new norms in the space of $(l+m)$-fold tensor products. For $p, q \in \mathbf{R}$, we define

$$|\, \kappa \,|_{K,A;l,m;p,q}^2 = \sum_{\mathbf{i},\mathbf{j}} |\, \langle \kappa,\, e(\mathbf{j}) \otimes e(\mathbf{i}) \rangle \,|^2 \,|\, e(\mathbf{j}) \,|_{K,p}^2 \,|\, e(\mathbf{i}) \,|_{A,q}^2, \qquad \kappa \in ({E_{\mathbf{C}}}^{\otimes(l+m)})^*,$$

Note that $|\, \kappa \,|_{A,A;l,m;p,p} = |\, \kappa \,|_{A,p}$. Moreover, for any $p, q, r \in \mathbf{R}$ it holds that

$$|\, \kappa \otimes_m f \,|_{K,A;l,n;q,r} \le |\, \kappa \,|_{K,A;l,m;q,-p} \,|\, f \,|_{A,A;n,m;r,p}\,.$$

In particular, for any $p \in \mathbf{R}$ and $q \ge 0$ it holds that

$$|\, \kappa \otimes_m f \,|_{K,p} \le |\kappa|_{K,A;l,m;p,-q} |f|_{K,A;n,m;p,q},$$

Lemma 10 *For any $p \in \mathbf{R}$ there exists $q \ge 0$ such that*

$$|\xi|_{K,p} \le |\xi|_{A,q}, \qquad \xi \in E_{\mathbf{C}}.$$

Proof. For any $p \in \mathbf{R}$, $E_{\mathbf{C}} \hookrightarrow D_{p,\mathbf{C}}$ is continuous. Therefore, there exist $C \ge 0$ and $q' \ge 0$ such that

$$|\, \xi \,|_{K,p} \le C \,|\, \xi \,|_{A,q'} \le C\rho^{q-q'} \,|\, \xi \,|_{A,q}, \qquad \xi \in E_{\mathbf{C}}, \quad q \ge q',$$

where $\rho = \|A^{-1}\|_{\mathrm{OP}} = 1/2$. Hence for a sufficiently large $q \ge 0$ we have $|\, \xi \,|_{K,p} \le |\, \xi \,|_{A,q}$, $\xi \in E_{\mathbf{C}}$. ■

Lemma 11 *For each $\kappa \in D_{p,\mathbf{C}}^{\otimes l} \otimes (E_{\mathbf{C}}^{\otimes m})^*$, the $\mathbf{C}$-valued function Θ given as in (5.3) satisfies condition (ii) in Theorem 8.*

Proof. By applying Lemma 10, we obtain that for any $r \geq 0$ with $|\kappa|_{K,A;l,m;p,-r} < \infty$ there exists $p' \geq p \vee r$ such that

$$\begin{aligned}
\left|(\kappa \otimes_m \xi^{\otimes m}) \otimes \xi^{\otimes n}\right|_{K,p} &\leq |\kappa|_{K,A;l,m;p,-r} |\xi|_{A,r}^m |\xi|_{K,p}^n \\
&\leq |\kappa|_{K,A;l,m;p,-r} |\xi|_{p'}^{m+n} \\
&\leq |\kappa|_{K,A;l,m;p,-r} \rho^{(m+n)(q-p')} |\xi|_q^{m+n},
\end{aligned}$$

where $q \geq p'$. Therefore, by direct computation, we have

$$|\Theta(\xi, K^p \cdot)|_{\mathcal{E}^2(\nu)}^2 \leq \sum_{n=0}^{\infty} \frac{(n+l)!\alpha(n+l)}{n!^2} |\kappa|_{K,A;l,m;p,-r}^2 \rho^{2(m+n)(q-p')} |\xi|_q^{2(m+n)}.$$

On the other hand, by Lemma 1, we have

$$\begin{aligned}
\frac{(n+l)!\alpha(n+l)}{n!^2} &= \frac{(n+l)!(n+m)!\alpha(n+l)}{n!^2(n+m)!} \\
&\leq \frac{2^{2n+l+m} n!^2 l! m! C_{2\alpha}^{n+l} C_{3\alpha}^{n+m} \alpha(n+m)\alpha(l)}{n!^2(n+m)!} \\
&= \frac{2^{2n+l+m} l! m! C_{2\alpha}^{n+l} C_{3\alpha}^{n+m} \alpha(l)\alpha(n+m)}{(n+m)!}.
\end{aligned}$$

Therefore, for some $q \geq p'$ such that $(4C_{2\alpha})^n C_{3\alpha}^{n+m} 2^m \rho^{2(m+n)(q-p')} \leq 1$

$$\begin{aligned}
|\Theta(\xi, K^p \cdot)|_{\mathcal{E}^2(\nu)}^2 &\leq |\kappa|_{K,A;l,m;p,-r}^2 l! m! (2C_{2\alpha})^l \alpha(l) \sum_{n=0}^{\infty} \frac{\alpha(n+m)}{(n+m)!} |\xi|_q^{2(m+n)} \\
&= |\kappa|_{K,A;l,m;p,-r}^2 l! m! (2C_{2\alpha})^l \alpha(l) G_\alpha(|\xi|_q^2).
\end{aligned}$$

It follows the proof. ■

Since the $\mathbf{C}$-valued function Θ given as in (5.3) satisfies conditions (i) and (ii) in Theorem 8, there exists an operator $\Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{D}_{\alpha,p})$ such that

$$\widehat{\Xi}^\alpha(\xi, \eta) = \left\langle \kappa, \eta^{\otimes l} \otimes \xi^{\otimes m} \right\rangle \sum_{n=0}^{\infty} \frac{\sqrt{\alpha(n+l)}}{n!} \langle \xi, \eta \rangle^n.$$

This operator is called a *integral kernel operator* with kernel distribution κ and denoted by $\Xi_{l,m}(\kappa)$. For each $t \in \mathbf{R}$, the operators $a_t = \Xi_{0,1}(\delta_t)$ and $a_t^* = \Xi_{1,0}(\delta_t)$ are called the *annihilation operator* and *creation operator*, respectively.

6 Wick Exponential

For two white noise operators $\Xi_1, \Xi_2 \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*)$, by Theorem 7, there exists a unique operator $\Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*)$ such that

$$\widehat{\Xi}(\xi, \eta) = \widehat{\Xi}_1(\xi, \eta) \widehat{\Xi}_2(\xi, \eta) e^{-\langle \xi, \eta \rangle}, \qquad \xi, \eta \in E_{\mathbf{C}}, \tag{6.4}$$

see [7]. The operator Ξ defined in (6.4) is called the *Wick product* of Ξ_1 and Ξ_2, and is denoted by $\Xi = \Xi_1 \diamond \Xi_2$. We note some simple properties:

$$I \diamond \Xi = \Xi \diamond I = \Xi, \qquad (\Xi_1 \diamond \Xi_2) \diamond \Xi_3 = \Xi_1 \diamond (\Xi_2 \diamond \Xi_3),$$

$$(\Xi_1 \diamond \Xi_2)^* = \Xi_2^* \diamond \Xi_1^*, \qquad \Xi_1 \diamond \Xi_2 = \Xi_2 \diamond \Xi_1.$$

Namely, equipped with the Wick product $\mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*)$ becomes a commutative $*$-algebra. As for the annihilation and creation operators we have

$$a_s \diamond a_t = a_s a_t, \qquad a_s^* \diamond a_t = a_s^* a_t, \qquad a_s \diamond a_t^* = a_t^* a_s, \qquad a_s^* \diamond a_t^* = a_s^* a_t^*. \tag{6.5}$$

More generally, it holds that

$$a_{s_1}^* \cdots a_{s_l}^* \Xi a_{t_1} \cdots a_{t_m} = \Xi \diamond (a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m}), \qquad \Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*).$$

In fact, the Wick product is a unique bilinear map from $\mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*) \times \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*)$ into $\mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*)$ which is (i) separately continuous; (ii) associative; and (iii) satisfying (6.5).

Theorem 12 [7] *Let α and ω be two weight sequences and assume that their generating functions are related in such a way that*

$$G_\omega(t) = \exp \gamma\{G_\alpha(t) - 1\}, \tag{6.6}$$

where $\gamma > 0$ is a certain constant. Then for any $\Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*)$, $\operatorname{wexp} \Xi \in \mathcal{L}(\mathfrak{W}_\omega, \mathfrak{W}_\omega^*)$, where $\operatorname{wexp} \Xi$ is the Wick exponential of Ξ defined by

$$\operatorname{wexp} \Xi = \sum_{n=0}^{\infty} \frac{1}{n!} \Xi^{\diamond n}.$$

Let $\kappa \in (E_{\mathbf{C}}^{\otimes(l_1+m_1)})^*$ and $\lambda \in (E_{\mathbf{C}}^{\otimes(l_2+m_2)})^*$. Then the Wick product of two integral kernel operators $\Xi_{l_1,m_1}(\kappa)$ and $\Xi_{l_2,m_2}(\lambda)$ is given by

$$\Xi_{l_1,m_1}(\kappa) \diamond \Xi_{l_2,m_2}(\lambda) = \Xi_{l_1+l_2,m_1+m_2}(\kappa \circ \lambda),$$

where $\kappa \circ \lambda \in (E_{\mathbf{C}}^{\otimes(l_1+l_2+m_1+m_2)})^*$ is defined by

$$\begin{aligned} &\kappa \circ \lambda(s_1, \cdots, s_{l_1+l_2}, t_1, \cdots, t_{m_1+m_2}) \\ &\quad = \kappa \otimes \lambda(s_1, \cdots, s_{l_1}, t_1, \cdots, t_{m_1}, s_{l_1+1}, \cdots, s_{l_1+l_2}, t_{m_1+1}, \cdots, t_{m_1+m_2}). \end{aligned}$$

Moreover, for any $\kappa \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ we have

$$\Xi_{l,m}(\kappa)^{\diamond n} = \Xi_{ln,mn}(\kappa^{\circ n}).$$

Theorem 13 *Let $\kappa \in D_{-p,\mathbf{C}}^{\otimes l} \otimes (E_{\mathbf{C}}^{\otimes m})^*$ and let α be a weighted sequence satisfying that*

$$C_\alpha = \sup\left\{ \frac{(k+ln)!(mn+k)!}{n!^2 k!^2 \alpha(k+ln)\alpha(mn+k)}; k, n \geq 0 \right\} < \infty.$$

Then we have

$$\operatorname{wexp} \Xi_{l,m}(\kappa) \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{D}_{1/\alpha,-p}).$$

Proof. For any $\Xi \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}_\alpha^*)$, we have

$$\widehat{\mathrm{wexp}\,\Xi}^{1/\alpha}(\xi,\eta) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{\Xi^{\diamond n}}^{1/\alpha}(\xi,\eta).$$

On the other hand, we have

$$\begin{aligned}\widehat{\Xi_{l,m}(\kappa)^{\diamond n}}^{1/\alpha}(\xi,\eta) &= \widehat{\Xi_{ln,mn}(\kappa^{\circ n})}^{1/\alpha}(\xi,\eta)\\ &= \left\langle \kappa^{\circ n}, \eta^{\otimes ln} \otimes \xi^{\otimes mn} \right\rangle \sum_{k=0}^{\infty} \frac{1}{k!\sqrt{\alpha(k+ln)}} \left\langle \xi, \eta \right\rangle^k .\end{aligned}$$

Therefore,

$$\begin{aligned}\widehat{\mathrm{wexp}\,\Xi}^{1/\alpha}(\xi,\eta) &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{1}{n!k!\sqrt{\alpha(k+ln)}} \left\langle (\kappa^{\circ n} \otimes_{mn} \xi^{\otimes mn}) \otimes \xi^{\otimes k}, \eta^{\otimes(ln+k)} \right\rangle\\ &= \sum_{i=0}^{\infty} \left\langle \sum_{k+ln=i} \frac{1}{n!k!\sqrt{\alpha(i)}} (\kappa^{\circ n} \otimes_{mn} \xi^{\otimes mn}) \otimes \xi^{\otimes k}, \eta^{\otimes i} \right\rangle .\end{aligned}$$

Hence

$$\|\widehat{\mathrm{wexp}\,\Xi}^{1/\alpha}(\xi, K^{-p}\cdot)\|_{\mathcal{E}^2(\nu)}^2 = \sum_{i=0}^{\infty} \frac{i!}{\alpha(i)} \left| \sum_{k+ln=i} \frac{1}{n!k!} (\kappa^{\circ n} \otimes_{mn} \xi^{\otimes mn}) \otimes \xi^{\otimes k} \right|_{K,-p}^2 .$$

On the other hand, for any $q \geq 0$ with $|\kappa|_{K,A;l,m;-p,-q} < \infty$ we have

$$\begin{aligned}&\left| \sum_{k+ln=i} \frac{1}{n!k!} (\kappa^{\circ n} \otimes_{mn} \xi^{\otimes mn}) \otimes \xi^{\otimes k} \right|_{K,-p}^2\\ &\qquad \leq \left(\left[\frac{i}{l}\right] + 1 \right) \sum_{k+ln=i} \frac{1}{n!^2k!^2} |\kappa|_{K,A;l,m;-p,-q}^{2n} |\xi|_{A,q}^{2mn} |\xi|_{K,-p}^{2k}\\ &\qquad \leq (i+1) \sum_{k+ln=i} \frac{1}{n!^2k!^2} |\kappa|_{K,A;l,m;-p,-q}^{2n} |\xi|_{A,q'}^{2(mn+k)},\end{aligned}$$

where $q' \geq q$ such that $|\xi|_{K,-p} \leq |\xi|_{A,q'}$. Therefore, we have

$$\begin{aligned}&\left| \sum_{k+ln=i} \frac{1}{n!k!} (\kappa^{\circ n} \otimes_{mn} \xi^{\otimes mn}) \otimes \xi^{\otimes k} \right|_{K,-p}^2\\ &\qquad \leq (i+1) \sum_{k+ln=i} \frac{1}{n!^2k!^2} |\kappa|_{K,A;l,m;-p,-q}^{2n} \rho^{2s(mn+k)} |\xi|_{A,q'+s}^{2(mn+k)} .\end{aligned}$$

Since there exists $s \geq 0$ such that $(k+ln+1)|\kappa|_{K,A;l,m;-p,-q}^{2n} \rho^{2s(mn+k)} \leq 1$,

$$\left| \sum_{k+ln=i} \frac{1}{n!k!} (\kappa^{\circ n} \otimes_{mn} \xi^{\otimes mn}) \otimes \xi^{\otimes k} \right|_{K,-p}^2 \leq \sum_{k+ln=i} \frac{1}{n!^2k!^2} |\xi|_{A,q'+s}^{2(mn+k)}$$

Therefore, we have

$$\begin{aligned}\|\widehat{\mathrm{wexp}\,\Xi}^{1/\alpha}(\xi, K^{-p}\cdot)\|^2_{\mathcal{E}^2(\nu)} &\le \sum_{i=0}^{\infty} \frac{i!}{\alpha(i)} \sum_{k+ln=i} \frac{1}{n!^2 k!^2} |\xi|^{2(mn+k)}_{A,q'+s} \\ &\le C_\alpha \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{\alpha(mn+k)}{(mn+k)!} |\xi|^{2(mn+k)}_{A,q'+s} \\ &\le C_\alpha \sum_{i=0}^{\infty} (i+1)\rho^{2ti} \frac{\alpha(i)}{i!} |\xi|^{2i}_{A,q'+s+t}.\end{aligned}$$

Thus for any $t \ge 0$ with $(i+1)\rho^{2ti} \le 1$ we have

$$\|\widehat{\mathrm{wexp}\,\Xi}^{1/\alpha}(\xi, K^{-p}\cdot)\|^2_{\mathcal{E}^2(\nu)} \le C_\alpha G_\alpha(|\xi|^2_{A,q'+s+t}).$$

Thus, by Theorem 9, the proof follows. ■

7 Normal-Ordered Differential Equations

In this section, as an application of characterizations, we consider an equation of the form:

$$\frac{d\Xi}{dt} = L_t \diamond \Xi, \qquad \Xi(0) = I, \tag{7.7}$$

where $t \mapsto L_t \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}^*_\alpha)$ is continuous. Equation (7.7) is generally called a *normal-ordered differential equation.* Recall that the space $\mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}^*_\alpha)$ is closed under the Wick product. Hence, a formal solution to (7.7) is given by the Wick exponential:

$$\Xi_t = \mathrm{wexp}\left(\int_0^t L_s\, ds\right) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\int_0^t L_s\, ds\right)^{\diamond n}, \tag{7.8}$$

and our first task is to check its convergence in the sense of generalized operators.

Several studies of the convergence of Wick exponential can be fund in [31], see also [30]. As a general result, we have the following

Theorem 14 [7] *Let α and ω be two weight sequences and assume that their generating functions are related as in (6.6). If $t \mapsto L_t \in \mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{W}^*_\alpha)$ is continuous, the solution is given by (7.8) and lies in $\mathcal{L}(\mathfrak{W}_\omega, \mathfrak{W}^*_\omega)$.*

Assume that L_t is an integral kernel operator:

$$L_t = \Xi_{l,m}(\lambda_{l,m}(t)). \tag{7.9}$$

In that case, the map $t \mapsto \lambda_{l,m}(t) \in (E_{\mathbf{C}}^{\otimes(l+m)})^*$ is continuous, and so is

$$t \mapsto \kappa_{l,m}(t) \equiv \int_0^t \lambda_{l,m}(s)\, ds \in (E_{\mathbf{C}}^{\otimes(l+m)})^*.$$

Since the (formal) solution of (7.7) is given by

$$\Xi_t = \text{wexp}\,\Xi_{l,m}(\kappa_{l,m}(t)),$$

see (7.8), regularity properties of Ξ_t is described in terms of $\kappa_{l,m}(t)$ instead of $\lambda_{l,m}(t)$.

The following theorem is straightforward from Theorem 13.

Theorem 15 *Assume that L_t is given by*

$$L_t = \Xi_{l,m}(\lambda_{l,m}(t)), \qquad \kappa_{l,m}(t) \equiv \int_0^t \lambda_{l,m}(s)\,ds \in (D_{-p,\mathbf{C}})^{\otimes l} \otimes (E_{\mathbf{C}}^{\otimes m})^*.$$

Let α be a weighted sequence satisfying that

$$C_\alpha = \sup\left\{\frac{(k+ln)!(mn+k)!}{n!^2k!^2\alpha(k+ln)\alpha(mn+k)}; k,n \geq 0\right\} < \infty.$$

Then, the unique solution to (7.7) lies in $\mathcal{L}(\mathfrak{W}_\alpha, \mathfrak{D}_{1/\alpha,-p})$.

Lemma 16 *Let $l, m \geq 0$. Then*

(1) *If $0 \leq l+m \leq 2$, then $C_{\widetilde{\beta}} < \infty$ for any $0 < \beta < 1$.*

(2) *If $2 < l+m$, then $C_{\widetilde{\beta}} < \infty$ for any $1 - 2/(l+m) < \beta < 1$.*

Proof. Since $\widetilde{\beta}(n) = n!^\beta$, $n \geq 0$, $0 \leq \beta < 1$, we have

$$\begin{aligned}\frac{(k+ln)!(mn+k)!}{n!^2k!^2\widetilde{\beta}(k+ln)\widetilde{\beta}(mn+k)} &= \frac{(k+ln)!^{1-\beta}(mn+k)!^{1-\beta}}{n!^2k!^2}\\ &= \frac{\left((l+1)^{k+ln}(m+1)^{k+mn}k!^2n!^{l+m}\right)^{1-\beta}}{n!^2k!^2}\end{aligned}$$

Therefore, if $0 \leq l+m \leq 2$, then $2(1-\beta) < 2$ and $(l+m)(1-\beta) < 2$ for any $0 < \beta < 1$. Hence $C_{\widetilde{\beta}} < \infty$ for any $0 < \beta < 1$. It follows the proof of (1).

On the other hand, if $2 < l+m$, then $2(1-\beta) < 2$ and $(l+m)(1-\beta) < 2$ for any $1 - 2/(l+m) < \beta$. Hence $C_{\widetilde{\beta}} < \infty$ for any $1 - 2/(l+m) < \beta < 1$. It follows the proof of (2). ∎

By Theorem 15 and Lemma 16, the following is obvious

Proposition 17 *Assume that L_t is given by*

$$L_t = \Xi_{l,m}(\lambda_{l,m}(t)), \qquad \kappa_{l,m}(t) \equiv \int_0^t \lambda_{l,m}(s)\,ds \in (D_{-p,\mathbf{C}})^{\otimes l} \otimes (E_{\mathbf{C}}^{\otimes m})^*.$$

Then we have

(1) *If $0 \leq l+m \leq 2$, the unique solution to (7.7) lies in $\mathcal{L}((E)_\beta, \mathfrak{D}_{1/\widetilde{\beta},-p})$ for any $0 < \beta < 1$.*

(2) *If $2 < l+m$, the unique solution to (7.7) lies in $\mathcal{L}((E)_\beta, \mathfrak{D}_{1/\widetilde{\beta},-p})$ for any $1 - 2/(l+m) < \beta < 1$.*

Now, the study of applications of the characterizations to wide class of (white noise) differential equations is being in progress.

References

[1] N. Asai, I. Kubo and H.-H. Kuo: *General characterization theorems and intrinsic topologies in white noise analysis*, Hiroshima Math. J. **31** (2001), 299–330.

[2] N. Asai, I. Kubo and H.-H. Kuo: *Segal-Bargmann transforms of one-mode interacting Fock spaces associated with Gaussian and Poisson measures*, preprint, 2001.

[3] D. M. Chung, T. S. Chung and U. C. Ji: *A simple proof of analytic characterization theorem for operator symbols*, Bull. Korean Math. Soc. **34** (1997), 421–436.

[4] D. M. Chung, U. C. Ji and N. Obata: *Higher powers of quantum white noises in terms of integral kernel operators*, Infinite Dimen. Anal. Quantum Prob. **1** (1998), 533–559.

[5] D. M. Chung, U. C. Ji and N. Obata: *Normal-ordered white noise differential equations I: Existence of solutions as Fock space operators*, in "Trends in Contemporary Infinite Dimensional Analysis and Quantum Probability (L. Accardi et al. Eds.)," pp. 115–135, Istituto Italiano di Cuitura, Kyoto, 2000.

[6] D. M. Chung, U. C. Ji and N. Obata: *Normal-ordered white noise differential equations II: Regularity properties of solutions*, in "Prob. Theory and Math. Stat. (B. Grigelionis et al. Eds.)," pp. 157–174, VSP/TEV Ltd., 1999.

[7] D. M. Chung, U. C. Ji and N. Obata: *Quantum stochastic analysis via white noise operators in weighted Fock space*, to appear in Rev. Math. Phys. 2002.

[8] W. G. Cochran, H.-H. Kuo and A. Sengupta: *A new class of white noise generalized functions*, Infinite Dimen. Anal. Quantum Prob. **1** (1998), 43–67.

[9] R. Gannoun, R. Hachaichi, H. Ouerdiane and A. Rezgui: *Un théoreme de dualité entre espaces de fonctions holomorphes à croissance exponentielle*, J. Funct. Anal. **171** (2000), 1–14.

[10] M. Grothaus, Yu. G. Kondratiev and L. Streit: *Complex Gaussian analysis and the Bargmann–Segal space*, Methods of Funct. Anal. and Topology **3** (1997), 46–64.

[11] L. Gross and P. Malliavin: *Hall's transform and the Segal–Bargmann map*, in "Itô's Stochastic Calculus and Probability Theory (N. Ikeda, S. Watanabe, M. Fukushima and H. Kunita (Eds.)," pp. 73–116, Springer-Verlag, 1996.

[12] T. Hida: "Analysis of Brownian Functionals," Carleton Math. Lect. Notes 13, Carleton University, Ottawa, 1975.

[13] T. Hida: "Brownian Motion," Springer–Verlag, 1980.

[14] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit (eds.): "White noise: An Infinite Dimensional Calculus", Kluwer Academic Publishers, 1993.

[15] Z.-Y. Huang: *Quantum white noises – White noise approach to quantum stochastic calculus*, Nagoya Math. J. **129** (1993), 23–42.

[16] Z.-Y. Huang and S.-L. Luo: *Wick calculus of generalized operators and its applications to quantum stochastic calculus*, Infinite Dimen. Anal. Quantum Prob. **1** (1998), 455–466.

[17] R. L. Hudson and K. R. Parthasarathy: *Quantum Itô's formula and stochastic evolutions*, Commun. Math. Phys. **93** (1984), 301–323.

[18] U. C. Ji and N. Obata: *A Role of Bargmann-Segal spaces in characterization and expansion of operators on Fock space*, preprint, 2000.

[19] U. C. Ji and N. Obata: *A unified characterization theorem in white noise theory*, preprint, 2001.

[20] U. C. Ji, N. Obata and H. Ouerdiane: *Analytic characterization of generalized Fock space operators as two-variable entire functions with growth condition*, to appear in Infinite Dimen. Anal. Quantum Probab. Rel. Top.

[21] Yu. G. Kondratiev and L. Streit: *Spaces of white noise distributions: Constructions, descriptions, applications I*, Rep. Math. Phys. **33** (1993), 341–366.

[22] I. Kubo and S. Takenaka: *Calculus on Gaussian white noise I–IV*, Proc. Japan Acad. **56A** (1980), 376–380; 411–416; **57A** (1981), 433–437; **58A** (1982), 186–189.

[23] H.-H. Kuo: "White Noise Distribution Theory," CRC Press, 1996.

[24] Y.-J. Lee and H.-H. Shih: *The Segal-Bargmann transform for Lévy functionals*, J. Funct. Anal. **168** (1999), 46–83.

[25] P.-A. Meyer: "Quantum Probability for Probabilists," Lect. Notes in Math. **1538**, Springer–Verlag, 1993.

[26] N. Obata: *An analytic characterization of symbols of operators on white noise functionals*, J. Math. Soc. Japan 45 (1993), 421-445.

[27] N. Obata: " White noise calculus and Fock space," Lecture Notes in Math., Vol. 1577, Springer, Berlin/ Heidelberg/ New York, 1994.

[28] N. Obata: *Generalized quantum stochastic processes on Fock space*, Publ. RIMS **31** (1995), 667–702.

[29] N. Obata: *Integral kernel operators on Fock space – Generalizations and applications to quantum dynamics*, Acta Appl. Math. **47** (1997), 49–77.

[30] N. Obata: *Quantum stochastic differential equations in terms of quantum white noise*, Nonlinear Analysis, Theory, Methods and Applications **30** (1997), 279–290.

[31] N. Obata: *Wick product of white noise operators and quantum stochastic differential equations*, J. Math. Soc. Japan. **51** (1999), 613–641.

[32] K. R. Parthasarathy: "An Introduction to Quantum Stochastic Calculus," Birkhäuser, 1992.

[33] J. Potthoff and L. Streit: *A characterization of Hida distributions*, J. Funct. Anal. **101** (1991), 212–229.

[34] Y. Yokoi: *Simple setting for white noise calculus using Bargmann space and Gauss transform*, Hiroshima Math. J. **25** (1995), 97–121.